How can we get students to justify? We know that the benefits of justification are numerous: Teachers have found that engaging students in justification can help students deepen and retain mathematical knowledge, gain a greater sense of ownership over the material, and improve communication and representation skills (Staples, Bartlo, and Thanheiser 2012). Student engagement in a justification activity can also lead to more equitable learning outcomes among diverse student populations (Boaler and Staples 2008). However, enacting lessons that help support student justifications can be challenging. One difficulty in implementing such tasks is that they are inherently open-ended; it takes careful planning using multiple strategies to guide students toward the intended mathematical goal.
Recent curricular standards have called for a greater emphasis on sense making, argumentation, and justification. For example, the Common Core’s Standard for Mathematical Practice 3 (CCSSI 2010, p. 6) asks students to “construct viable arguments and critique the reasoning of others,” and NCTM’s Principles and Standards for School Mathematics encourages students of all grade levels to “develop and evaluate mathematical arguments and proofs” (NCTM 2000, p. 56). We focus on justification because it is less formal than a mathematical proof. A justification goes beyond showing what a student did: A justification should provide a convincing mathematical argument for why a statement is true.

To better understand how to support student justifications within middle school math classrooms, a team of researchers and twelve middle school teachers collaborated on the Justification and Argumentation: Growing Understanding of Algebraic Reasoning (JAGUAR) project. Within JAGUAR, we developed and refined strategies to promote student justifications. These strategies included the deliberate setup of the task, an explicit identification of specialized vocabulary, and careful listening to student thinking, thus creating many informal opportunities for students to exchange ideas and encouraging the use of multiple representations. We found that one key to a better implementation of justification tasks involved teachers “doing the math” as part of the lesson-planning process.

WHAT IS “DOING THE MATH”? Doing the Math was a part of intensive lesson planning in which teachers completed a detailed investigation of the mathematics in a task. Smith, Bill, and Hughes (2008) provided a lesson-planning protocol that aimed to help...
teachers maintain a high level of cognitive demand for a task. We saw “doing the math” as an important part of this type of lesson planning. Doing the math involved working through a problem and anticipating potential student justifications and misconceptions. It also helped us uncover the potential of a task, identify the task’s goals, understand the prior knowledge necessary to access the task, and anticipate solutions to better support students while teaching. In this article, we illustrate doing the math with the Hexagon task (Smith, Silver, and Stein 2005) and conclude with a discussion of the benefits for both students and teachers that we have experienced.

Hexagon Task: An Example
The Hexagon task is a good example of an open-ended problem-solving task accessible to middle school students. Students identify a pattern in a geometric figure and justify a generalization of that pattern for any length figure (see fig. 1). The task gives students opportunities to form connections between different representations and describe a mathematical relationship. It also allows for different solution strategies and encourages students to explain their reasoning.

Generating a Justification
Doing the math starts by solving the task and creating a justification. While working on the mechanical steps used to solve the problem, we tried to connect our problem-solving process to the mathematics of why our thinking is valid. Our goal was to create a robust justification that used mathematics to explain why the solution was correct.

We began the Hexagon task by creating a table that compared the figure number with the perimeter of the figure (see table 1). As the figure number went up by 1, the perimeter went up by 4. From the table, we generated a linear equation with a growth rate of 4 and an initial value of 2, leading to the equation \( P = 4F + 2 \).

Recognizing that a pattern existed in the table was an important insight, but for a complete justification, we needed to explain the mathematics behind why that relationship held in the table. Our next step was to connect the values in the table to the original figure. In figure 2, one student used arrows to connect the diagram to each piece of the equation. The growth rate of 4 came from the 4 exposed sides of each hexagon (2 top and 2 bottom), and the additional 2 came from the two “ends.”

Constructing Alternate Justifications
Once we worked through the task and created one justification, we revisited the problem to justify it in a different way. To generate ideas for alternate strategies, we tried to anticipate ways that students might think about the task, different problem-solving approaches, or different representations that students might use to justify their solution. For example, how would a justification look different if the student used a graph rather than a table to represent the problem? We worked
through the task using the manipulatives or technology that students might use to solve the task. For each strategy, our goal was to unpack and explore the mathematics of why the reasoning was mathematically correct.

For the hexagon task, the most basic strategy for justifying that the perimeter of the 25th figure was 102 was to draw the 25th figure and count the perimeter. This mathematical strategy, although valid, was inefficient and could not be generalized to find the perimeter of any figure. Once we identified this potential strategy, we went a step further to consider what we might do if a student were to use or present this strategy in our classroom. We might ask the student, for example, to look for shortcuts or patterns in how they counted the perimeter.

We also developed alternate algebraic expressions by thinking about the diagram of the hexagon train in different ways. For example, the 2 end hexagons each had 5 exposed sides and the interior hexagons had 4 sides, yielding an equation \( P = 10 + 4(N - 2) \), where \( P \) was the perimeter and \( N \) represented the number of hexagons in the chain (see fig. 3).

Although this equation was different from the one found during our initial investigation, it was mathematically equivalent. By thinking about the task differently, we realized that the task contained opportunities to explore multiple big math ideas, such as identifying patterns and comparing equivalent expressions and their representations. By doing the math, we discovered that this single task could be used for multiple math learning goals.

While imagining alternative ways to justify a task, we also tried to think of possible incorrect strategies. For example, a student might identify a growth rate of 4 and develop the formula \( P = 4F \), failing to account for the “+2.”

In figure 4, a student clearly explained how the pattern grew by 4 each time but did not account for the +2.

To help students improve their formula, we might ask them to test their formula with examples: “Check this formula with smaller examples to make sure that the method will work.” Testing examples can help a student see that the formula does not work, but we wanted our students to understand why, so we redirected them to the diagram and asked them to identify how the diagram was growing and

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Fig. 3 The hexagons on each end had a perimeter of 5; each interior hexagon had a perimeter of 4. If there were \( N \) total hexagons, there were \((N - 2)\) interior hexagons. Thus, the total perimeter, \( P \), was given by \( P = 10 + 4(N - 2) \).

![Diagram of hexagon chain](image1)

**Fig. 4** This student explored the growth rate of 4. In (a), a drawing showed that a new hexagon added 4 new sides; in (b) the student wrote that the previous end was “pushed” outward to become the new end.

![Diagram of hexagon growth](image2)

As the number of hexagons increase by one the perimeter increases by four.
where the pieces of their formula connected. We asked, “We multiply by 4 because whenever you add a hexagon, it adds 4 units. Where are the 4 sides being added in the picture? Does that count all the sides of the perimeter?”

Another possible incorrect strategy included the misapplication of proportional reasoning. In the past, our students had incorrectly applied proportional reasoning to nonproportional relationships, which, research indicated, was common (see Lannin 2005). The perimeter of the 5th figure is 22, so students may conjecture that the perimeter of the 10th figure would be 44 because it was twice as big. This conjecture was incorrect because the relationship between the perimeter and figure was not proportional. It was true that when two trains of length 5 were considered independently, each train had a perimeter of 22; however, when the trains met to form a train of length 10, 2 sides were pressed together and should not be counted in the new perimeter (see fig. 5).

When doing the math, we had time to reflect on how we might confront this possible misconception. Like the previous misconception, we could address this misconception by asking the student to draw the 10th figure and count the perimeter. Giving a counterexample shows students that they were wrong, but it might not help extend their thinking. To understand why it did not work, it was more convincing for a student to see what happened when two trains were pressed together using manipulatives or a diagram.

**Goal Setting**

One of the reasons for doing the math was to make us aware of the mathematical potential of the task. Initially, we saw that the task focused on identifying an algebraic pattern and connecting it to a rule, but it was only by digging deeper that we saw potential for the exploration of equivalent expressions or proportional reasoning. Once we identified the big math ideas, we used school, district, and state content standards to help us focus the big math ideas of the task into an exact mathematical goal for the lesson. Part of identifying the mathematical goal of the lesson was to identify which skills, practices, or understandings from this lesson should be carried forward to future mathematical lessons. For this task, our goal was to justify the relationship between the independent variable (figure number) and dependent variable (perimeter) in a table, graph, equation, and diagram, which corresponded to the sixth-grade Common Core State Standard 6.EE.C.9, “represent and analyze quantitative relationships between dependent and independent variables” (CCSSI 2010, p. 41). Doing the math helped to make us aware of the mathematical potential of the task; state standards helped narrow the focus.

Once we identified our goal, we refined the task as written. Doing the math helped us identify the areas of the problem that had the potential for justification, and those parts could be emphasized in implementation. Also, portions of the task that might not pertain closely to the mathematical goal could be eliminated. By doing the math, we gained a deeper understanding of not only the potential for each part of the task but also the possible solutions that could emerge from exploring those questions.

**Prior Knowledge and Mathematical Vocabulary**

Doing the math also helped us determine what prior knowledge students will need to be able to access the task. By completing the task multiple times using multiple strategies, we gained a better understanding of the different types of mathematical tools, such as a table or a diagram, that students might use to form a justification. In addition, it helped us identify the relevant mathematical vocabulary. For example, to launch the task, we reminded students of what independent and dependent variables were to frame their thinking about the problem. We also made sure that a common understanding existed about how to find the perimeter of a hexagon train. Spending time working with the task gave us opportunities to become more comfortable using specialized terms and helped us speak more precisely with our students.
BENEFITS AND DRAWBACKS

One drawback is obvious: It takes time to do detailed mathematical planning for the task. It can seem overwhelming to do the math for every lesson every time, but not every lesson needs such detailed work each time. We started by identifying just a few examples in our curriculum that had the potential for justification: Open-ended tasks that allowed for multiple solution strategies. We started by doing the math for a few of these examples. We realized that planning helped us lead a more efficient lesson, and now we do it for all our “big” lessons.

Benefits in the Classroom

Doing the math as part of planning helped us become better guides for our students. We began to have a clearer idea of the justifications that we wanted to see from our students and the mathematics that they will need to be able to get to those justifications. We were more prepared to interpret student thinking because we had already considered many different strategies. When we saw a particular idea emerging from a student, we were already prepared to ask questions to guide them toward a mathematically valid justification.

For example, if a student was having trouble getting started, we suggested using a table to generate a rule. If a student had generated a rule but was having difficulty justifying why the rule held, we directed his or her attention to the parts of the diagram related to that particular rule: “Where in the diagram do you see +2?” or “How many exposed sides are on the end hexagons? How many exposed sides are on the middle hexagons?” If a student was attempting to use a proportional reasoning strategy, we asked a student to draw a diagram of each of the smaller trains and explain what happens to the perimeter when those smaller trains are “smashed” together to form one longer train (see fig. 6).

Even if unanticipated strategies emerged in the class (which will always happen, no matter the amount of planning), we were more prepared to interpret the student’s mathematical spontaneous thinking because of the time we had spent exploring the mathematical underpinnings of the task. For example, one group of students devised the rule:

\[ P = (6x) - ((x - 1)2) \]

(see fig. 7).

Although we had not anticipated this formula, our work with...
the faulty proportional reasoning strategy helped us make sense of this group’s formula. The “6x” represented the total number of sides (x hexagons with 6 sides each). There were (x – 1) places where 2 sides were pressed together, thus, 2(x – 1) interior sides needed to be removed. By visualizing the removal of interior sides, we could make sense of the students’ idea rather than simply verifying algebraically that the formula 6x – 2(x – 1) is equivalent to 4x + 2.

Doing the math also helped us lead our classroom more effectively. The launch of a task was more intentional because we reminded students of the exact concepts and language needed. Wrap-up at the end of class was more focused because the exact mathematical goals had been set ahead of time.

We had already thought through how we would like to select and sequence student justifications for the whole class, and we were better able to connect those justifications to the content goals for the lesson. Although our class was more efficient, we still made sure our students were the ones doing the mathematical thinking.

We gave them opportunities to make mistakes, evaluated the validity of their strategies, and analyzed the strategies of others. By doing the math, we were more prepared to provide the tools that students needed to approach the task and the language that was necessary to communicate their ideas.

**Increased Confidence**

We felt more confident when leading a lesson after having done the math because we were secure in our knowledge of the material. We were less likely to be sidetracked in off-topic explorations because we had already explored alternate avenues and discovered those what were the most fruitful. If students ran into difficulty, we had preplanned questions that could help direct or challenge them.

Students also felt more confident when approaching a lesson requiring justification. They had a clear idea of the tools needed to explore the task, and they had a clear idea of the expectations for a justification. If they found themselves at a dead-end, they knew that we could help redirect them by giving them something new to consider. Our students had greater confidence sharing their mathematical thinking when they understood what was being asked and had been reminded of the mathematical language to use to express themselves.

**Benefits over Time**

There were benefits to repeatedly “doing the math” using the same lesson. Each time we revisited a task, we gained a deeper insight into the mathematical connections in the material that we had not previously seen. Doing the math helped us understand the mathematical structure of the content more deeply and helped us make connections between content areas. As we saw more connections, we could better articulate and refine the goal of the lesson to connect with past and future lessons. The process of doing the math became more efficient as we gained experience thinking through different justifications. We saw “doing the math” as an important part of lesson planning to support open-ended tasks, particularly justification tasks.

**REFERENCES**

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