Modeling with Mathematics

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Introduction. The Arizona Mathematics Partnership (AMP) is committed to helping our partner schools provide strong programs in mathematics. Part of this commitment is aimed at delivering high quality professional development that is aligned with the Common Core State Standards in Mathematics (i.e., Arizona's College and Career Ready Standards) and the Standards for Mathematical Practice (http://www.corestandards.org/Math/Practice). One of the mathematical practices highlighted by the Common Core – MP4 Model with Mathematics – is an area that is specifically addressed by the AMP program and is the area of focus for this paper.

What is "Model with Mathematics"? First, it is important to distinguish between the various meanings of *modeling*. Modeling can be thought of as an instructional strategy used to teach mathematics. The TAP rubric embraces the notion of "I do – You do – We do." From our work with AMP teachers, we have discovered that many of them interpret this statement to mean "teachers demonstrate an example of a problem for students (I do), the class completes similar problems (we do), and then students are given more of the same problems to do independently for homework (you do). This is a traditional mode of instruction in mathematics, but we are trying to move teachers away from this approach and towards the kind of mathematics classroom that the authors of the Common Core envisioned, and that research in math education has shown to be successful.

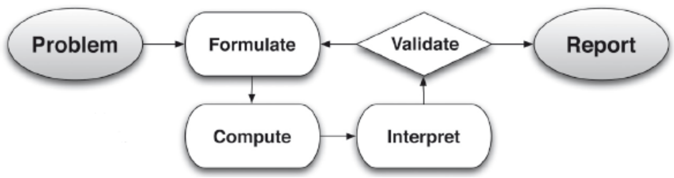
The Common Core's intention of *modeling*, as described in the paragraph below, is more about the mathematics and less about the instructional strategy used to teach the mathematics.

CCSS.Math.Practice.MP4 Model with mathematics.

> Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

The Common Core describes modeling as linking "classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions." (page 72, http://www.corestandards.org/Math/Content/HSM). Simply stated, modeling with mathematics refers to using mathematical thinking in a creative way to tackle non-standard, non-routine, complex problems. Just as art students (and sculptors) might use clay to model an object of interest, math students (and mathematicians) model with mathematics to make sense of a problem, analyze relationships and draw conclusions.

Modeling with mathematics entails a *process* undertaken by students, typically working in teams. It is *not* something learned by purposeful mimicking; nor is it best developed by "I do – You do – We do" techniques. Rather, modeling with mathematics is a *learn-by-doing* and *acquire-by-experience process* that grows student capacity to solve complex, practical problems arising in everyday life and in the workplace. The Common Core pictures modeling as a cycle that involves students in the acts of formulating, computing, interpreting, validating and reporting.



Modeling with mathematics develops students' proficiency in all aspects of problem solving: apply the mathematics they know; make assumptions; make approximations to simplify a complicated situation; identify relevant quantities; use tools such as diagrams, tables, graphs, flowcharts, formulas; analyze relationships; draw conclusions; reflect on whether the results make sense; improve their model; communicate results and solutions.

Model with Math: An Example of Fraction Multiplication. For teaching multiplication of fractions using traditional methods, a typical instructional strategy is to give students "rules" and demonstrate "procedures." Modeling with mathematics, however, offers an alternative approach to merely memorizing rules and procedures (that may be forgotten, confused, misunderstood or misapplied). When students perform multiplication of fractions, we want them to give meaning to what they are doing and to know why they are doing it; ultimately, we want them to develop and internalize the traditional algorithms. Modeling with mathematics provides a way to foster sense-making and deep understanding.

The example below provides classroom teachers with a concrete example of how to use modeling with mathematics to help students develop rich and well-connected understandings as they learn multiplication of fractions. This example gives a picture of what modeling with mathematics can look like for teaching a core content skill through developing meaning of the mathematics. In fact, any of the core content areas can be taught using a "model with math" design. To the extent that modeling with mathematics becomes a routine process for students as they learn the "basic skills," tackling more elaborate real-world problems with this process also becomes more routine and natural.

An Example of "Modeling with Mathematics"
Understanding Multiplication of Fractions

Dr. Scott Adamson, Chandler-Gilbert Community College

Unfortunately, it is common for students to "learn" to multiply fractions by being told the rule. Often, they are told to "multiply straight across" or "just multiply the numerators and the denominators." For example,

$$\frac{7}{8}\cdot\frac{3}{4}=\frac{7\cdot 3}{8\cdot 4}=\frac{21}{32}$$

This style of teaching leads to several misconceptions.

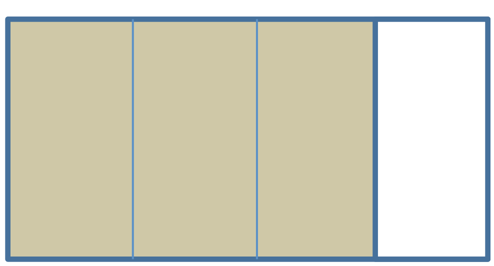
- Students learn to view a fraction as consisting of two whole numbers rather than learning to develop a multiplicative understanding of the idea of a fraction.
- Students fail to develop number sense. Why does seven-eighths times three-fourths produce a result that is greater than one-half but less than one?
- Students often try to use the same procedure when adding fractions. That is, they incorrectly add the numerators and add the denominators when adding two fractions.

When void of meaning and understanding, in the mind of a student, the multiplication of two fractions algorithm becomes just another of a long list of procedures to follow – procedures that may or may not make sense to the student.

Modeling with mathematics provides a means by which students can make sense of multiplication of fractions. That is, as students engage in the modeling process, they develop meanings and understandings from which the desired procedure may emerge in a way that makes sense to students. Consider the following situation.

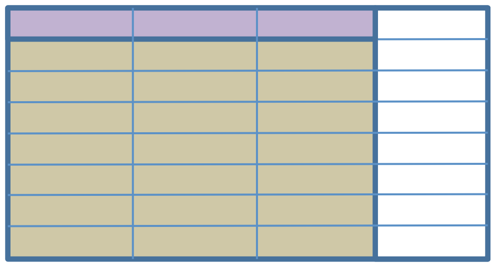
> *Terry noticed that there was 3/4 of a rectangular cake left after the party. He notices that 1/8 of the remaining cake had all of the frosting taken from it. What part of the original cake remaining still has frosting?*

Students may be encouraged to model the situation by representing the fact that 3/4 of a rectangular cake is left after the party.



Note that the rectangle has been cut up into 4 equal parts. Each part represents 1/4 of the whole rectangle. Three of these parts are shaded. That is, 3 copies of 1/4 are shaded. We say 3 copies of 1/4, or 3 one-fourths, or simply 3/4.

Now, 1/8 of *this* amount had all the frosting removed from it. To represent this idea, we can cut the remaining cake into 8 equal parts as shown. One-eighth of the remaining three-fourths of the cake is represented by the top row shaded in purple.



Using this model, students can see the portion of the remaining cake (shaded brown in this model) that still has frosting. They can determine that there are $7 \cdot 3 = 21$ such pieces. When compared to the entire cake (remember that we originally started with 3/4 of a whole cake), this portion that still has frosting is $\frac{21}{32}$ of the entire cake.

Using this method of modeling situations, students may come to understand that seven-eighths of the remaining cake (which is three-fourths of the whole cake) still has frosting. That is,

$$\frac{7}{8} \cdot \frac{3}{4}.$$

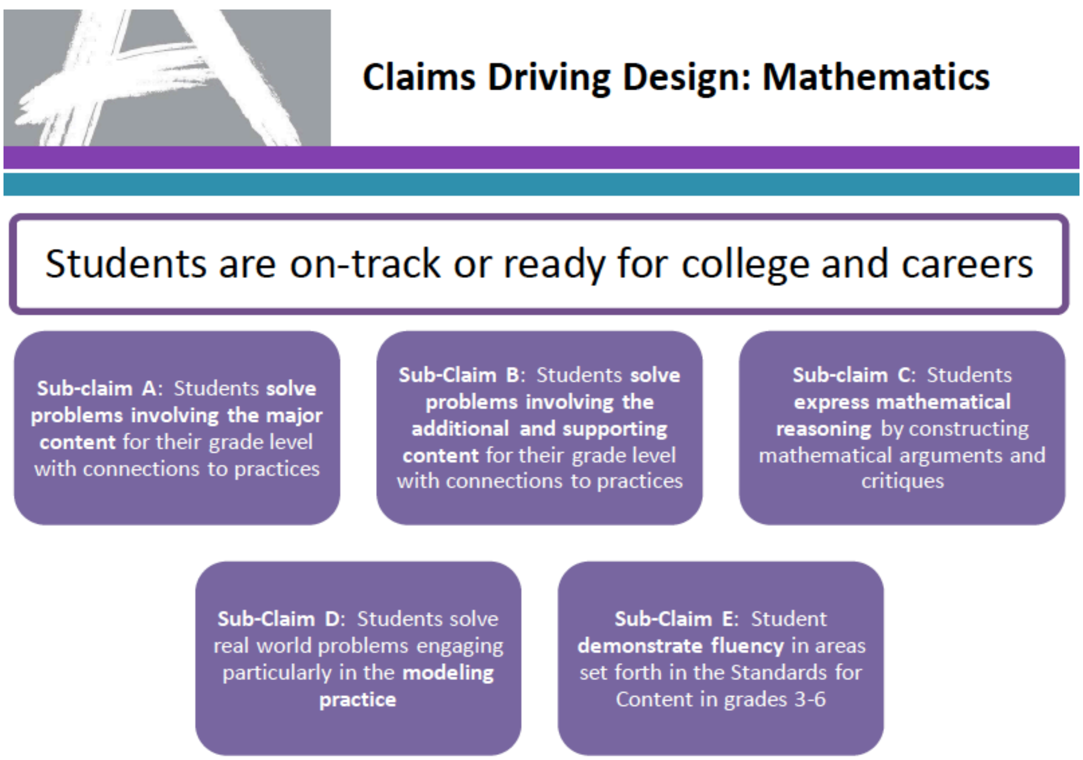
With repeated reasoning (SMP #8), students may come to realize that the product of the numerators ($7 \cdot 3 = 21$) represents the number of pieces of cake under consideration in the problem (in this case, the amount of cake that still has frosting). The product of the denominator ($8 \cdot 4 = 32$) represents the total number of pieces of cake. This leads to the traditional algorithm for multiplying fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

With careful instruction, students can make sense of the product of two fractions by thinking about taking, for example, seven-eighths of a copy of three-fourths. They have a powerful way of thinking about fractions multiplicatively. They will have the opportunity to develop a mental model for this multiplication. That is, they may learn to imagine what seven-eighths of a copy of three-fourths might look like.

When students are afforded the opportunity to model with mathematics, the traditional algorithms can emerge from student thinking. The algorithms then are part of a well-connected network of understanding of ideas and mental images. When this is the case, students are able to develop computational fluency and precision.

Assessment Claims. Both the Partnership for Assessment of Readiness for College and Careers (PARCC) and the Smarter Balanced Assessment Consortium (SBAC) stipulate claims for assessing the practice of modeling with mathematics. As such, students will need to have developed this ability from their past mathematics experiences in order to demonstrate proficiency on the new assessment. These assessment claims are being used as a design framework for developing assessment items, and in particular, modeling is one of the 5 sub-claims used for item development. For the PARCC assessment, modeling is specifically identified in Sub-Claim D: "Students solve real word problems engaging particularly in the modeling practice" (http://www.ieanea.org/media/2013/02/PARCC-Math-Assessment.pdf).



Concluding Remarks. Modeling with mathematics is an area in which classroom teachers need strong backing from their school administrators. Modeling with mathematics looks very different from traditional problem solving activities that math teachers are already familiar with. Understanding this difference and developing a "model with mathematics" teaching practice are important professional development goals that the AMP project wants to support among our partner schools.

Suggested Reading

Of Practical Interest:

Bostic, J. D. (2013). Model-eliciting activities for teaching mathematics. *Mathematics Teaching in the Middle School*, *18*(5), 262-266.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. Kelly & R. Lesh (Eds.), *Research design in mathematics and science education.* (pp. 591-646). Lawrence Erlbaum Associates, Mahwah, NJ.

Magiera, M. T. (2013). Eliciting activities: A home run. *Mathematics Teaching in the Middle School*, *18*(6), 348-355.

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Szydlik, S. D. (2009). The problem with the snack. *Mathematics Teacher*, *103*(1), 18-25.

Of Research Interest:

Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education*, *34*(2), 110-136.

English, L. D. (2003). Reconciling theory, research, and practice: A models and modeling perspective. *Educational Studies in Mathematics*, *54*(2&3), 225-248.

English, L. D. (2006). Mathematical modeling in the primary school: Children's construction of a consumer guide. *Educational Studies in Mathematics*, *63*(3), 303-323.

Huntley, M. A., Rasmussen, C. L., Villarubi, R. S., Sangtong, J., & Fey, J. T. (2000). Effects of *standards*-based mathematics education: A study of the Core-Plus mathematics project algebra and functions strand. *Journal for Research in Mathematics Education*, *31*(3), 328-361.

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Lesh, R., Middleton, J. A., Caylor, E., & Gupta, S. (2008). A science need: Designing tasks to engage students in modeling complex data. *Educational Studies in Mathematics*, *68*(2), 113-130.

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Smith, E., Haarer, S., & Confrey, J. (1997). Seeking diversity in mathematics education: Mathematical modeling in the practice of biologists and mathematicians. *Science and Education*, *6*(5), 441-472.

Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in Mathematics*, *63*(1), 89-11.